Math 565: Functional Analysis
Lecture 15

Uniform boundedness principle (Banach–Steinhaus theorem). Let X, Y be normed vector spaces. Let $\mathcal{T} \subseteq B(X,Y) :=$ bdd linear maps.

(a) If $\sup_{T \in \mathcal{T}} \|Tx\| < \infty$ for nonmeagre many $x \in X$ (i.e. $\{x \in X : \sup_{T \in \mathcal{T}} \|Tx\| < \infty\}$ is nonmeagre), then $\sup_{T \in \mathcal{T}} \|T\| < \infty$.

(b) If X is Banach and $\sup_{T \in \mathcal{T}} \|Tx\| < \infty$ for all $x \in X$, then $\sup_{T \in \mathcal{T}} \|T\| < \infty$.

Proof. (b) is an instance of (a), so we only prove (a). Let $D := \{x \in X : \sup_{T \in \mathcal{T}} \|Tx\| < \infty\} = \bigcup_{n \in \mathbb{N}} D_n$, where $D_n := \{x \in X : \sup_{T \in \mathcal{T}} \|Tx\| \le n\} = \bigcap_{T \in \mathcal{T}} \{x \in X : \|Tx\| \le n\}$ is closed. By the hypothesis, D is nonmeagre, hence $\exists n \in \mathbb{N}$ with D_n being somewhere dense, so $\mathrm{int}(D_n) \ne \emptyset$ because D_n is closed. Hence $\exists \overline{B}_\varepsilon(x_0) \subseteq D_n$ for some $\varepsilon > 0$, $x_0 \in D_n$, so $\forall T \in \mathcal{T}$ $\forall x \in \overline{B}_\varepsilon(0)$, we have:

$$\|Tx\| \le \|Tx_0\| + \|T(x_0 - x)\| \le n + n = 2n,$$

which implies that $\forall T \in \mathcal{T}$, $\|T\| \le 2n/\varepsilon$. $\square$

Remark. The technique of localizing to some $n \in \mathbb{N}$, as in this proof, also in the proof the open mapping theorem, is called localization in the context of Baire spaces.

Before stating an interesting corollary, recall that for normed vector spaces X, Y, if Y is Banach then $B(X,Y)$ is also Banach, i.e. it is "closed" in norm topology. It also makes sense to consider the pointwise convergence topology on $B(X,Y)$, i.e. just the product topology on $B(X,Y) \subseteq Y^X$. When is $B(X,Y)$ closed under pointwise convergence, i.e. $B(X,Y) \subseteq Y^X$ is closed?

Cor. Let X, Y be normed vector spaces. If X is Banach, then $B(X,Y)$ is closed under pointwise convergence, i.e. if $(T_n) \subseteq B(X,Y)$ and for each $x \in X$, $Tx := \lim_{n \to \infty} T_n x$ exists, then $T \in B(X,Y)$.

Proof. The linearity of T is just because limits are linear. For bddness of T, note that for each $x \in X$, the set $\{\|T_n x\|\}_{n\in\mathbb{N}}$ is bdd because $(\|T_n x\|)$ converges to $\|Tx\|$ and convergent sequences are bdd. By the uniform bddness principle, $\{\|T_n\|\}_{n\in\mathbb{N}}$ is bdd and $\|T\| \leq \sup_{n\in\mathbb{N}} \|T_n\|$ because $\forall x \in X$, $\|Tx\| = \lim_n \|T_n x\| \leq \sup_n \|T_n\| \cdot \|x\|$. □

Topological vector spaces.

Definitions and examples.

Normed vector spaces are nice but there are important examples of vector spaces equipped with natural topologies which are not generated by a norm.

Examples. (a) $L^1_{loc}(\mathbb{R}^d, \lambda)$; (b) $C(X)$ where X is lcH, with the uniform-on-compact topology; (c) $L^0(X,\mu) :=$ all μ-measurable functions mod μ-null for a probability space (X,μ); (d) the space $C^\infty([0,1])$ of infinitely differentiable functions on $[0,1]$.

Def. A *topological vector space* over $\mathbb{K} := \mathbb{R}, \mathbb{C}$ is a vector space X over $\mathbb{K}$ equipped with a topology making addition $(x,y) \mapsto x+y : X \times X \to X$ and scalar multiplication $(\alpha, x) \mapsto \alpha x : \mathbb{K} \times X \to X$ continuous (in particular, inversion $x \mapsto -x : X \to X$ is continuous). Such a space X is called *locally convex* if its topology admits a basis consisting of convex sets.

Most natural topologies on vector spaces are generated by a family $\{p_i\}_{i\in I}$ of semi-norms, i.e. by the *balls* of these semi-norms: $B^i_r(x_0) := \{x \in X : p_i(x_0 - x) < r\}$, $i \in I$ and $r \geq 0$.

Prop. Let X be a vector space and $\{p_i\}_{i\in I}$ be a family of semi-norms on X generating a topology on X.

(a) This topology makes X into a locally convex top. vector space.
(b) For each $x_0 \in X$, the finite intersections of $B^i_{1/n}(x_0)$, $i \in I$, $n \in \mathbb{N}^+$, form a nghbhd basis at x_0.
In particular, if I is ctbl then X is 1^{st} ctbl, in fact, metrizable.
(c) For any sequence (more generally, net) $(x_n) \subseteq X$, $x_n \to x \iff p_i(x - x_n) \to 0 \ \forall i \in I$.
(d) X is Hausdorff $\iff$ for each $0 \neq x \in X$ $\exists\, i \in I$ with $p_i(x) \neq 0$.

Proof. (a) Follows from triangle inequality for semi-norms
(b) This is just a general straightforward fact from topology that if a collection Σ of open sets generates the topology then finite intersections of sets in Σ form a basis for the topology.
The metrizability is left as a HW exercise.
(c) $\Rightarrow$. This is by the definition of $x_n \to x$ using p_i-balls at x as open neighbourhoods.
$\Leftarrow$. Let $\mathcal{U} \ni x$ be open, and by (a), $\mathcal{U} \supseteq \bigcap_{j<k} B^{i_j}_{r_j}(x)$ for some $k \in \mathbb{N}$, $i_j \in I$, $r_j > 0$. For each $j<k$, $p_{i_j}(x - x_n) \to 0$ implies $\forall^\infty_n \ x_n \in B^{i_j}_{r_j}(x)$. Since $k < \infty$, $\forall j<k \ \forall^\infty_n \iff \forall^\infty_n \forall j<k$, so $\forall^\infty_n \forall j<k$ $x_n \in B^{i_j}_{r_j}(x)$, hence $x_n \in \bigcap_{j<k} B^{i_j}_{r_j}(x) \subseteq \mathcal{U}$.
(d) $\Leftarrow$. If $x \neq y$, $x - y \neq 0$, so $\delta := p_i(x-y) > 0$ for some $i \in I$. Then $B^i_{\delta/2}(x)$ and $B^i_{\delta/2}(y)$ are disjoint open neighbourhoods of x and y, resp.
$\Rightarrow$. Let $x \neq 0$, so Hausdorffness in particular gives an open $\mathcal{U} \ni 0$ with $x \notin \mathcal{U}$. By (a), $\mathcal{U} \supseteq \bigcap_{j<k} B^{i_j}_{r_j}(0)$ so $B^{i_j}_{r_j}(0) \not\ni x$ for at least one $j<k$, i.e. $p_{i_j}(x) \geq r_j > 0$. □

Examples.
(a) $L^1_{loc}(\mathbb{R}^d, \lambda) :=$ the space of locally Lebesgue integrable functions, i.e. λ-measurable functions $f: \mathbb{R}^d \to \mathbb{C}$ s.t. $\| f|_{B_n} \|_1 < \infty$ for all balls B_n at 0 of radius $n \in \mathbb{N}$, where two functions are identified if they are $=$ a.e.
The natural top on this space is generated by the seminorms
$$p_n(f) := \| f|_{B_n} \|_1, \ n \in \mathbb{N}.$$
In particular, $L^1_{loc}(\mathbb{R}^d, \lambda)$ is locally convex metrizable top. vec space.
This can be generalized to locally finite Borel measures on any top space X, i.e. a Borel measure μ on X s.t. X admits a cover by open sets of finite measure.

(6) Let (X,μ) be a prob. space and let $L^0(X,\mu)$ denote the space of μ-measurable functions $f: X \to \mathbb{C}$ mod null. Recall that a sequence $(f_n) \in L^0(X,\mu)$ converges to $f \in L^0(X,\mu)$ in measure if $\forall \alpha > 0$, $p_\alpha(f-f_n) := \mu(\{x \in X : |f(x)-f_n(x)| \geq \alpha\}) \to 0$ as $n \to \infty$. Thus, the topology of this notion of convergence is the one generated by the "quasi-semi-norms" $p_\alpha(f) := \mu(\{x \in X : |f(x)| \geq \alpha\})$ for all $\alpha > 0$; equivalently all $\alpha > 0$ rational. These p_α are not semi-norms: $p_2(1+1) = 1 \nleq 0 = p_2(1) + p_2(1)$, but they satisfy a "quasi-triangle inequality": $p_{\alpha+\beta}(f+g) \leq p_\alpha(f) + p_\beta(g)$ for all $\alpha, \beta > 0$ and $f, g \in L^0(X,\mu)$. It follows from this inequality that these p_α make X into a topological vector space. This space is Hausdorff: if $f \neq 0$ mod null, then $\exists n \in \mathbb{N}^+$ s.t. $\{|f| \geq 1/n\}$ has positive measure, so $p_{1/n}(f) > 0$. It is also 1^{st} ctbl since α can range over $\mathbb{Q}^+$. In fact, $L^0(X,\mu)$ is metrizable by the metric $d(f,g) := \inf_{\delta > 0} (\mu(p_\delta(f-g)) + \delta)$, see HW 5 Q5 of Math 564, Fall 2025.